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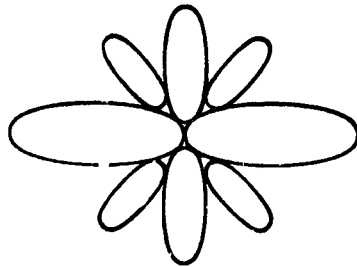
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"SAMPLE" VALUES POSSIBLY OUTLIERS**

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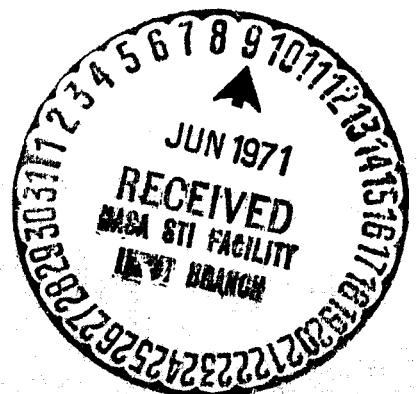
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NONPARAMETRIC ESTIMATION OF MEAN AND VARIANCE WHEN A FEW

"SAMPLE" VALUES POSSIBLY OUTLIERS

John E. Walsh

Southern Methodist University*

ABSTRACT

The data (continuous) are n independent observations that are believed to be a random sample. The possibility exists, however, that as many as J of the largest observations, and as many as k of the smallest observations, are outliers. That is, these observations are from populations that are different from the population yielding the other observations (which number at least $n-J-K$). The interest is in obtaining suitable estimates for the mean and variance of the population yielding the other observations. J and K are given and relatively small, with both $\leq 2n^A$, where A is specified and $\leq 1/4$. When the population yielding the other observations is continuous, has moments of all orders, and is well-behaved in some other ways, estimates are developed that are unbiased if terms of order $n^{-1+A+2\epsilon}$ are neglected. Here, ϵ can be arbitrarily small but is positive.

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INTRODUCTION AND RESULTS

The data are n independent observations from continuous univariate populations. These observations are believed to be a random sample and estimates are desired for the population mean and variance. However, there is the possibility that as many as J of the largest observations and as many as K of the smallest observations are from populations that differ from the population yielding the other observations. Then, the interest is in obtaining suitable estimates for the mean μ and the variance σ^2 of the population yielding the random sample (of size at least $n-J-K$) that consists of the other observations. The values of J and K are given and relatively small. Specifically, $0 \leq J, K \leq 2n^A$, where A is given and such that $0 \leq A \leq 1/4$.

Let the order statistics of the n observations be denoted by

$$x(1) < x(2) < \dots < x(n-1) < x(n).$$

Then, $x(1), \dots, x(k)$ and $x(n+1-j), \dots, x(n)$ are from populations that differ from the population yielding $x(k+1), \dots, x(n-j)$, which constitute a random sample of size $n-j-k$. Here, $j = 0$ implies that none of the largest observations are from differing populations and $k = 0$ implies that none of the smallest observations are from differing populations. The values of j and k are unknown but satisfy $j \leq J$ and $k \leq K$.

The properties stated for the estimates presented do not hold in general. These estimates are not applicable unless n is at least moderately large and the population yielding the random sample of size n satisfies some conditions (at least approximately). Besides being continuous, this population should have finite moments of all orders and should

have a density function that is analytic and nonzero throughout the range of possible values. A more exact statement of these conditions is given in the Derivations section.

The estimates could be stated in many ways. The statement given here uses all of $x(k+1), \dots, x(n-j)$ with equal weighting. These are the only observations that are known to be from the population with mean μ and variance σ^2 .

The estimate of μ is denoted by $\bar{x}(J,K)$ and the estimate of σ^2 is $S(J,K)$, where $\bar{x}(J,K)$ equals

$$(n-J-K)^{-1} [x(K+1) + x(K+2) + \dots + x(n-J)]$$

and $S(J,K)$ equals

$$(n-J-K-1)^{-1} [x(K+1)^2 + \dots + x(n-J)^2] \\ - [(n-J-K)/(n-J-K-1)] \bar{x}(J,K)^2.$$

These estimates have the properties

$$E[\bar{x}(J,K)] = \mu + O(n^{-1+A+\epsilon}),$$

$$E[S(J,K)] = \sigma^2 + O(n^{-1+A+2\epsilon}),$$

$$\text{Var}[\bar{x}(J,K)] = \sigma^2/n + o(n^{-1}),$$

$$\text{Var}[S(J,K)] = O(n^{-1}),$$

where $\epsilon > 0$ is a fixed but arbitrarily small constant. It is to be remembered that $1/4$ is the largest possible value for A .

The next, and final, section contains an outline of the derivations for the properties of $\bar{x}(J,K)$ and $S(J,K)$.

OUTLINE OF DERIVATIONS

The relationships occurring in the derivations are similar to those arising in ref. 1. For brevity, much of the verification is only outlined, with referral to ref. 1 for more details.

The basic approach is to state $\bar{x}(J,K)$ and $S(J,K)$ in terms of $x(k+1), \dots, x(n-j)$, which is a random sample from the population considered, plus additional terms. Then, expressions whose expectations are μ and σ^2 , respectively, can be identified and the additional terms are shown to be unimportant for n sufficiently large.

Some notation is introduced first. The mean of the sample of size $n-j-k$ is denoted by $\bar{x}(j,k)$ and is obtained from the expression for $\bar{x}(J,K)$ by letting $J = j$ and $K = k$. The arithmetic average of the order statistics $x(k+1), \dots, x(K), x(n-J+1), \dots, x(n-j)$ is denoted by y and the arithmetic average of the squares of these order statistics is represented by y^2 .

Let $F(x)$ be the cumulative distribution function of the population yielding $x(k+1), \dots, x(n-j)$, and let $x^{(t)}(z)$, for $t = 0, 1, 2, \dots$, be defined by

$$F[x^{(0)}(z)], \quad x^{(t)}(z) = d^t x^{(0)}(z) / dz^t.$$

The more exact conditions on $F(x)$ are: $x^{(0)}(z)$ can be expanded in Taylor series about each of the values $z = (k+1)/(n-j-k), \dots, K/(n-j-k), (n-J+1)/(n-j-k), \dots, (n-j)/n-j-k$ and, for each series, $\int_0^1 [x^{(0)}(z)]^b dz$ can be evaluated using term by term integration ($b=1, \dots, 4$). Also, the magnitude of $z^t x^{(t)}(z)$ is at most $O(1)$ with respect to n for these values

of z , ($t=1,2,\dots$), and the $x^{(0)}(z)$ are at most $O(n^\epsilon)$, where $\epsilon > 0$ is arbitrarily small but a fixed constant. For $t = 2,3,\dots$, the magnitude of $z^t x^{(t)}(z)$ is at most $o(1)$ for these values of z .

These conditions (taken from ref. 1) are not very restrictive for practical situations involving continuous populations. The first part justifies some expansions that are used. The magnitude relationships for the $x^{(0)}(z)$ are motivated by the consideration that this is the case when all the population moments exist. The relationships involving the $x^{(t)}(z)$ for $t \geq 1$ hold for nearly all continuous populations of practical interest.

The expectation of $\bar{x}(J,K)$ is considered first. The value of $\bar{x}(J,K)$ can be expressed as

$$[(n-j-k)/(n-J-K)]\bar{x}(j,k) + [(J+K-j-k)/(n-J-K)]y$$

Thus,

$$E[\bar{x}(J,K)] = \mu + O(n^{-1+A+\epsilon}),$$

since

$$E[\bar{x}(j,k)] = \mu, \quad E(y) = O[(n-j-k)^\epsilon]$$

and j,k,J,K are $O(n^A)$.

Next, consider the variance of $\bar{x}(J,K)$. By a method very similar to that used in ref. 1 (for the variance of m_x considered there), the variance of $\bar{x}(J,K)$ is found to be $\sigma^2/n + o(n^{-1})$. The principal use of this result is in evaluation of the expectation of $S(J,K)$. Another result for this purpose is

$$E(Z^2) = \text{Var}(Z) + [E(Z)]^2,$$

which applies, in particular, when Z is an order statistic. From the stated conditions, and material in ref. 1,

$$E(Z^2) = O[(n-j-k)^{2\epsilon}]$$

when Z is any of $x(k+1), \dots, x(K), x(n-J+1), \dots, x(n-j)$.

Now, consider the expectation of $S(J,K)$. The value of $S(J,K)$ can be expressed as

$$\begin{aligned} & [(n-j-k-1)/(n-J-K-1)] (n-j-k-1)^{-1} [x(k+1)^2 + \dots + x(n-j)^2] \\ & - [(J+K-j-K)/(n-J-K-1)] \bar{y}^2 \\ & - [(n-J-K)/(n-J-K-1)] \bar{x}(J,K)^2. \end{aligned}$$

Thus, $E[S(J,K)]$ equals

$$\begin{aligned} & [(n-j-k-1)/(n-J-K-1)] (\sigma^2 + \mu^2) - (J+K-j-k) (n-J-K-1)^{-1} O[(n-j-k)^{2\epsilon}] \\ & - [(n-J-K)/(n-J-K-1)] [\sigma^2/n + O(n^{-1}) + \mu^2 + O(n^{-1+A+\epsilon})] \\ & = \sigma^2 + O(n^{-1+A+2\epsilon}). \end{aligned}$$

The fact that $\text{Var}[S(J,K)]$ is $O(n^{-1})$ is verified by a method very similar to that used in ref. 1 (for the variance of S_x^2 considered there).

REFERENCE

1. John E. Walsh, "Nonparametric mean and variance estimation from truncated data," Skandinavisk Aktuarietidskrift, Vol 41 (1958), pp. 125-130.

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